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The mass term in non-Abelian gauge field dynamics on matrix D-branes and T-duality in the σ-model approach

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ABSTRACT: The formal extension of the T-duality rules for open strings from Abelian to non-Abelian gauge field background leads in a well known manner to the notion of matrix valued D-brane position. The application of this concept to the non-Abelian gauge field RG β-function of the corresponding σ-model yields a mass term in the gauge field dynamics on the matrix D-brane. The direct calculation in a corresponding D-brane model does *not* yield such a mass term, if the Dirichlet boundary condition is implemented as a constraint on the integrand in the defining functional integral. However, the mass term arises in the direct calculation for a D-brane model with dynamically realized boundary condition.

KEYWORDS: D-branes, M(atrix) Theories, String Duality, Conformal Field Models in String Theory.

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Contents

1. Introduction

T-duality for open strings interchanges Neumann and Dirichlet boundary conditions [1, 2]. If in a more general setting the string couples with its free ends to an Abelian gauge field, the boundary condition stating the balance of the normal derivative of the world sheet embedding and the Lorentz force

$$\partial_{\mathbf{n}} X^\mu \; + \; 2\pi\alpha' F^\mu{}_\nu \partial_{\mathbf{t}} X^\nu = 0$$

turns under the T-duality for the $i^{\rm th}$ coordinate $\partial_{\mathbf{n}} X^i \to \partial_{\mathbf{t}} X^i$ (**n** and **t** denote normal and tangential, respectively) into the condition

$$X^i = 2\pi\alpha' A^i(X^M)\,,$$

if the gauge field A_μ depends only on a subset X^M of the coordinates $X^\mu = (X^i, X^M)$. Thus in the dual description the string endpoints have to move on the manifold, the D-brane, defined by the gauge fields in the i-directions.

The formal extension of this recipe to non-Abelian gauge fields [3] leads to the notion of matrix valued D-brane positions which plays a crucial role in the heuristic motivation of M(atrix) theory [4] and the emergence of non-commutative geometric structures in string theory [5].

In this paper we will comment on the issue of a σ-model description of the T-dual model in the case of non-Abelian gauge fields including its quantum corrections and the issue of realizing the boundary condition either as an external one or a dynamical one. By external realization we mean a condition on the fields which are integrated over in the functional integral, by dynamical realization we have in mind conditions arising as part of the stationarity condition of the action. Both kinds of realization are equivalent on the classical level, but may lead to different quantum theories.

The formal manipulations of the functional integral for the σ-model, describing the motion of a closed string in nontrivial target space fields (G, B, Φ) related to its massless excitations, which yield Buscher's duality rules [6], have been extended to a σ-model on a manifold with boundary which in addition couples to a gauge field in ref. [7], see also [8]. There a dual model with externally realized Dirichlet boundary conditions is generated. The RG β-functions of this model have been calculated in ref.[9] already. The resulting conformal invariance conditions are equivalent to the stationarity conditions of the Born-Infeld action dimensionally reduced to the D-brane [9]. The Born-Infeld action is, up to the dimensional reduction, form invariant under the T-duality transformation [8]. Therefore, at least up to the implemented level of accuracy, the gauge field part of the naive T-duality rule is compatible with renormalisation.[1]

With the help of an one-dimensional auxiliary ζ-field formalism [11, 12] resolving the path ordering prescription for the Wilson loop we were able in [7, 13] to extend the treatment to the case of non-Abelian gauge fields. In this way the notion of matrix valued brane position got a special realization suitable for practical calculations. Postponing the ζ-integration until the very end, at intermediate stages the non-Abelian gauge field appears sandwiched between $\bar{\zeta}$ and ζ only

$$\bar{\zeta}_a(s)(A_\mu(X(z(s)))_{ab}\zeta_b(s)\,.$$

Hence the only difference to the Abelian situation is an explicit dependence on the parameter s, parametrising the boundary of the string world sheet. The gauge field and brane position RG β-functions have been calculated in lowest order of α' in [13].

Our only focus in this paper concerns the mass term for non-Abelian gauge field dynamics on matrix D-branes which arises by direct application of the naive T-duality rule to the β-function of the dual partner, i.e. the usual open string with free ends moving in target space non-Abelian gauge field background. Such a term does *not* appear in the direct calculation of the D-brane model [13, 14]. Thus at this state of affairs the non-Abelian gauge field part of the naive T-duality transformation seems to be incompatible with renormalisation at lowest order, already. Before sketching the plan of the paper we will recall the formal duality argument giving rise to the mass term [14].

The lowest order gauge field β-function for the string with free ends is[2][15, 16]

$$\tilde{\beta}_\lambda^{(\tilde{A})} = -\alpha'\ \eta^{\mu\nu}\ \tilde{D}_\mu \tilde{F}_{\nu\lambda}\,. \tag{1.1}$$

[1]See also the discussion in the second ref. of [7] and the discussion of T-duality and renormalisation for the closed string fields [10].

[2]We denote all quantities in this model with a tilde. The untilded quantities refer to the D-brane model.

If e.g. $\partial_1 A_\mu = 0$ we get, using $X^1 = f(X^M) = -2\pi\alpha'\tilde{A}_1(\tilde{X}^M)$, $\eta^{11} = -1$ and $\tilde{X}^M = X^M$, $A_1 = 0$, $A_M = \tilde{A}_M$,

$$\tilde{\beta}_L^{(\tilde{A})} = -\alpha' \, \eta^{MN} \, D_M F_{NL} \; + \; \frac{i}{4\pi^2\alpha'}\,[f, D_L f]\,,$$
$$\tilde{\beta}_1^{(\tilde{A})} = \frac{1}{2\pi}\eta^{MN} \, D_M D_N f\,. \tag{1.2}$$

The second term in $\tilde{\beta}_\lambda^{(\tilde{A})}$ turns out as a standard mass term, at least in the special D-brane configuration of diagonal and constant f, i.e. $f_{ab} = f_a\delta_{ab}$, $\partial_M f_a = 0$, which describes planar parallel D-brane copies at position $X^1 = f_a$, $a = 1,\ldots,n$. The equation of motion obtained from $\tilde{\beta}_L^{(A)} = 0$ then is

$$(D^M \, F_{ML})_{ab} + \left(\frac{1}{2\pi\alpha'}\right)^2 (f_a - f_b)^2 (A_L)_{ab} = 0\,. \tag{1.3}$$

As expected the mass is proportional to the separation of the D-brane copies.

The bulk of the paper is organised as follows. To keep things as clear as possible we restrict ourselves to the case of trivial closed string background fields and consider T-duality in one direction (X^1) only. The calculations crucially depend on the ζ auxiliary formalism. To become familiar with this calculus we start in section 2 with the derivation of the well known variational formulae for the standard Wilson loop [17]. We do this in a rather explicit way to amplify the step crucially also in the later modified applications: The use of the $\bar{\zeta}$, ζ equation of motion in correlation functions including contact terms.

To present a simple check of our results in [13, 14] free of the subtleties connected with the covariant expansion of the imposed boundary constraint, we use in section 3 the simplification due to flat target space to integrate the functional δ-function. Things become even simpler if one then considers constant f only. But also in this very transparent situation no mass term appears.

In section 4 we switch to dynamically realized Dirichlet conditions. The corresponding model is formulated quantising the action obtained in our paper [7] by a canonical transformation of the free end model. As it will turn out, this model yields a gauge field β-function with mass term. In the final section we summarise and interprete our result. In particular it will be stressed that the reason for the discrepancy of both versions is strongly related to a partial interchange of the functional integration over the string position with that over the auxiliary ζ-field.

2. Derivation of the standard variational formula for the Wilson loop in ζ-language

The Wilson loop as a functional of a closed path $X^\mu(s)$, $0 \le s \le 1$, $X^\mu(0) = X^\mu(1)$ defined by

$$\mathcal{W}[X] = \mathrm{tr}\ \mathrm{P}\exp\left(i\int_0^1 A_\mu(X(s))\dot{X}^\mu \, ds\right) \tag{2.1}$$

can be expressed by the following functional integral over an one-dimensional auxiliary field $\zeta,\ \bar\zeta$ with propagator $\langle\bar\zeta_a(t)\zeta_b(s)\rangle = \delta_{ab}\Theta(s-t)$ [11, 12]

$$\mathcal{W}[X] = \int D\bar\zeta D\zeta\ \bar\zeta_a(0)\zeta_a(1)\ e^{iS_0[\zeta,\bar\zeta]}\ W[X,\zeta,\bar\zeta]\,. \tag{2.2}$$

S_0 and W are defined by

$$S_0[\zeta,\bar\zeta] = i\int_0^1 \bar\zeta(s)\dot\zeta(s)ds\ +\ i\bar\zeta(0)\zeta(0) \tag{2.3}$$

and

$$W[X,\zeta,\bar\zeta] = \exp\left(i\int_0^1 \bar\zeta A_\mu(X(s))\zeta(s)\ \dot X^\mu ds\right). \tag{2.4}$$

Perhaps it is useful to stress that due to the sum over the index a, realizing the trace, there is no dependence on the choice of the point on the closed path from which we count s. Varying with respect to X and performing one partial integration we get

$$\frac{\delta W}{\delta X^\mu(t)} = iW\left(\bar\zeta(\partial_\mu A_\lambda - \partial_\lambda A_\mu)\zeta\dot X^\lambda(t) - \dot{\bar\zeta}A_\mu\zeta - \bar\zeta A_\mu\dot\zeta\right).$$

Now the use of the equation of motion for ζ and $\bar\zeta$ just produces the commutator term, necessary to complete the full non-Abelian field strength. To verify this we note that for the full ζ-action

$$S[\zeta,\bar\zeta] = S_0[\zeta,\bar\zeta] - i\log W = S_0[\zeta,\bar\zeta] + \int_0^1 \bar\zeta A_\mu\zeta\dot X^\mu ds \tag{2.5}$$

one has (the corresponding eq. for $\dot{\bar\zeta}$ is obvious)

$$\dot\zeta = iA_\mu\zeta\dot X^\mu - i\frac{\delta S}{\delta\bar\zeta(s)}\,. \tag{2.6}$$

The argument is then completed by the application of

$$\int D\bar\zeta D\zeta\ \bar\zeta_a(0)\zeta_a(1)\ e^{iS[\zeta,\bar\zeta]}\ h(\zeta(t),\bar\zeta(t))\ \frac{\delta S}{\delta\bar\zeta(t)} = 0\,, \tag{2.7}$$

which is valid for an arbitrary polynomial h. Altogether we arrive at[3]

$$\frac{\delta W}{\delta X^\mu(t)} = iW\ \bar\zeta F_{\mu\lambda}\zeta\ \dot X^\lambda\,, \tag{2.8}$$

i.e.

$$\frac{\delta\mathcal{W}}{\delta X^\mu(t)} = i\ \mathrm{tr}\ \mathrm{P}\left(e^{i\int_0^1 A_\nu\dot X^\nu ds}F_{\mu\lambda}\dot X^\lambda(t)\right). \tag{2.9}$$

[3]Here and in the following we understand all equations for W and its derivatives up to terms vanishing under the $\zeta,\bar\zeta$-integration.

The second functional derivative becomes

$$\frac{\delta^2 W}{\delta X^\mu(t)\delta X^\nu(t')} = iW\ \delta(t-t')\ \bar\zeta\partial_\nu F_{\mu\lambda}\zeta\ \dot X^\lambda(t)\ +\ iW\ \dot\delta(t-t')\ \bar\zeta F_{\mu\nu}\zeta$$
$$-W\ \bar\zeta F_{\mu\lambda}\zeta\ \dot X^\lambda(t)\ \left(\bar\zeta(\partial_\nu A_\kappa - \partial_\kappa A_\nu)\zeta\ \dot X^\kappa(t') - \dot{\bar\zeta} A_\nu\zeta - \bar\zeta A_\nu\dot\zeta(t')\right). \quad (2.10)$$

In the bilocal (t,t')-term we again want to use the ζ-equations of motion to get the Abelian part of the field strength at t' completed by the commutator term to the full non-Abelian field strength. But now there is a second factor at parameter value t which is responsible for the appearance of a contact term $\propto \delta(t-t')$ which just covariantizes the derivative of $F_{\mu\lambda}$ in the first term of the r.h.s. of (2.10). The underlying general formula is (k and h polynomials in ζ, $\bar\zeta$)

$$\int D\bar\zeta D\zeta\ \bar\zeta_a(0)\zeta_a(1)\ e^{iS[\zeta,\bar\zeta]}\times$$
$$\times\left(ih(\zeta(t),\bar\zeta(t))\ \frac{\delta S}{\delta\bar\zeta(t)}\ k(\zeta(t'),\bar\zeta(t'))\ +\ \delta(t-t')\ h((\zeta(t),\bar\zeta(t))\ \frac{\partial k}{\partial\bar\zeta(t)}\right) = 0. \quad (2.11)$$

Applying this to eq.(2.10) we get

$$\frac{\delta^2 W}{\delta X^\mu(t)\delta X^\nu(t')} = iW\ \delta(t-t')\ \bar\zeta D_\nu F_{\mu\lambda}\zeta\ \dot X^\lambda(t)\ +\ iW\ \dot\delta(t-t')\ \bar\zeta F_{\mu\nu}\zeta$$
$$-W\ \bar\zeta F_{\mu\lambda}\zeta\ \dot X^\lambda(t)\ \bar\zeta F_{\nu\kappa}\zeta\ \dot X^\kappa(t')\,. \quad (2.12)$$

The transition to $\frac{\delta^2\ \mathcal{W}}{\delta X^\mu(t)\delta X^\nu(t')}$ is, similar to the pair (2.8),(2.9), obvious and will not be written down. Finally we note that (2.8) and (2.12) can be summarised in

$$W[X+Y,\zeta,\bar\zeta] = W[X,\zeta,\bar\zeta]\times$$
$$\times\ \exp\left(i\int_0^1\bar\zeta(F_{\mu\lambda}\dot X^\lambda Y^\mu + \frac{1}{2}D_\mu F_{\nu\lambda}\dot X^\lambda Y^\mu Y^\nu + \frac{1}{2}F_{\mu\nu}Y^\mu\dot Y^\nu)\zeta ds + O(Y^3)\right). \quad (2.13)$$

3. β-functions in the case of external Dirichlet boundary condition

We consider the simplest case of only one coordinate of type i and call this X^1. Then the D-brane is defined by $X^1 = f(X)$ with $\partial_1 f = 0$. $A_M(X)$, $\partial_1 A_M(X) = 0$ describes the gauge field on the brane. Both A and f are matrix-valued. The string world sheet M is parametrized by the two-dimensional coordinate z, the boundary ∂M in z-space is described by $z(s)$. The partition function for the model we will discuss in this section is given by

$$\hat{\mathcal{Z}}[A,f] = \int DX^\mu D\bar\zeta D\zeta\ \bar\zeta_a(0)\zeta_a(1)\ e^{iS_0[\zeta,\bar\zeta]}\ \delta_{\partial M}(X^1 - \bar\zeta f(X)\zeta)\times$$
$$\times\exp\left(\frac{i}{4\pi\alpha'}\int_M d^2z(\partial X^N\partial X_N + \partial X^1\partial X_1) + i\int_{\partial M}\bar\zeta A_N\zeta\ \dot X^N ds\right). \quad (3.1)$$

It is a special case of the model discussed in ref.[13]. In a first step we can perform the X^1-integration

$$
\begin{aligned}
I[\bar\zeta f\zeta] &\equiv \int DX^1 \exp\left(\frac{i}{4\pi\alpha'}\int_M d^2z\ \partial X^1\partial X_1\right)\delta_{\partial M}(X^1-\bar\zeta f(X)\zeta)\\
&= (\det\partial^2)^{-\frac{1}{2}}\exp\left(\frac{i}{4\pi\alpha'}\int_{\partial M}\bar X^1\partial_{\mathbf{n}}\bar X_1 ds\right). \qquad (3.2)
\end{aligned}
$$

The functional determinant of the Laplacian is taken within the space of functions approaching the value zero at the boundary. In the above equation $\bar X^1$ is the unique solution[4] of the boundary value problem

$$
\partial^2\bar X^1 = 0 \quad \text{in } M\,, \qquad\qquad \bar X^1 = \bar\zeta f\zeta \quad \text{on } \partial M\,. \qquad (3.3)
$$

Up to now there was no need to specify any periodicity condition for $\zeta,\bar\zeta$. However, since the boundary value problem is well posed for $\bar\zeta f\zeta(1) = \bar\zeta f\zeta(0)$ only, we now choose ζ and $\bar\zeta$ anti-periodic [12]. Then the propagator in the fundamental interval $0<s<1$ is still given by $\delta_{ab}\Theta(s-t)$. The periodic choice for $\zeta,\bar\zeta$ would lead to an saw-toothed propagator which no longer could organise the path ordering.

The solution of (3.3) can be represented with some kernel $p(z,s)$ fixed by the geometry of M as (For a circle the corresponding integral is the familiar Poisson integral.)

$$
\bar X^1(z) = \int_0^1 ds\ p(z,s)\bar\zeta(s)f(X^M(z(s)))\zeta(s)\,. \qquad (3.4)
$$

Denoting the boundary value of $\partial_{\mathbf{n}}p(z,t)$ by $q(s,t)$ we get (skipping the irrelevant determinant factor)

$$
I[\bar\zeta f\zeta] = \exp\left(\frac{-i}{4\pi\alpha'}\int_{\partial M}\int_{\partial M}\bar\zeta f\zeta(s)\ \bar\zeta f\zeta(t)\ q(s,t)ds dt\right). \qquad (3.5)
$$

Insertion of this result into (3.1) implies

$$
\begin{aligned}
\hat{\mathcal{Z}}[A,f] = \int DX^M D\bar\zeta D\zeta\ \bar\zeta_a(0)\zeta_a(1)\ e^{iS_0[\zeta,\bar\zeta]\times}&\\
\times\ \exp\left(\frac{i}{4\pi\alpha'}\int_M d^2z\ \partial X^N\partial X_N\right)\cdot\hat W[X,\zeta,\bar\zeta]\,,& \qquad (3.6)
\end{aligned}
$$

where we introduced

$$
\hat W[X,\zeta,\bar\zeta] = \exp\left(i\int_{\partial M}\bar\zeta A_N\zeta\dot X^N ds\ -\ \frac{i}{4\pi\alpha'}\int_{\partial M}\int_{\partial M}\bar\zeta f\zeta(s)\bar\zeta f\zeta(t)q(s,t)dsdt\right). \qquad (3.7)
$$

It is very crucial that the modified Wilson loop $\hat W[X,\zeta,\bar\zeta]$ is a functional of $X^N(z(s))$ only.

Up to now $f(X)$ was an arbitrary matrix-valued function of the coordinates X^N. To keep the variational formulae for $\hat W[X,\zeta,\bar\zeta]$ as simple as possible, we restrict ourselves

[4] We switch to Euclidean two-dimensional world volume.

in the rest of this section to *constant* f. Then the bilocal term in $\hat{W}[X,\zeta,\bar{\zeta}]$ has influence on the $\zeta,\bar{\zeta}$ equations of motion only. The total action for $\zeta,\bar{\zeta}$ instead of S from section 2 is

$$\begin{aligned}\hat{S}[\zeta,\bar{\zeta}] &= S_0[\zeta,\bar{\zeta}] - i\log\hat{W}\\ &= S_0[\zeta,\bar{\zeta}] + \int_0^1 \bar{\zeta}A_N\zeta\dot{X}^N ds - \\ &\quad - \frac{1}{4\pi\alpha'}\int_{\partial M}\int_{\partial M}\bar{\zeta}f\zeta(s)\;\bar{\zeta}f\zeta(t)\;q(s,t)dsdt\,, \end{aligned} \tag{3.8}$$

leading to

$$\dot{\zeta}(t) = iA_N\zeta\dot{X}^N(t) - \frac{i}{2\pi\alpha'}f\zeta(t)\int_0^1 \bar{\zeta}f\zeta(s)q(t,s)ds \;-\; i\frac{\delta\hat{S}}{\delta\bar{\zeta}(s)}\,. \tag{3.9}$$

Note that the integral in the last line just represents $\partial_{\mathbf{n}}X^1$. Therefore the equation of motion obtained by setting $\frac{\delta\hat{S}}{\delta\bar{\zeta}(s)}$ to zero coincides with the formal dualization of (2.6). Repeating now with the modified quantities the steps presented in section 2 we get

$$\hat{W}[X+Y,\zeta,\bar{\zeta}] = \hat{W}[X,\zeta,\bar{\zeta}]\;\exp(i\hat{\Omega}[X,Y,\zeta,\bar{\zeta}])\,, \tag{3.10}$$

with

$$\begin{aligned}&\hat{\Omega}[X,Y,\zeta,\bar{\zeta}] = O(Y^3)\\ &+\int_0^1\Big\{\Big(\bar{\zeta}F_{ML}\zeta(s)\dot{X}^L - \frac{i}{2\pi\alpha'}\bar{\zeta}[f,A_M]\zeta(s)\int_0^1\bar{\zeta}f\zeta(t)\;q(s,t)dt\Big)Y^M\\ &+\frac{1}{2}\bar{\zeta}D_MF_{NK}\zeta(s)\;\dot{X}^KY^MY^N \;+\; \frac{1}{2}\bar{\zeta}F_{MN}\zeta(s)\;Y^M\dot{Y}^N\\ &-\frac{i}{4\pi\alpha'}\bar{\zeta}D_M[f,A_N]\zeta(s)\;Y^MY^N\int_0^1\bar{\zeta}f\zeta(t)q(s,t)dt\Big\}ds\\ &+\frac{1}{4\pi\alpha'}\int_0^1\int_0^1\bar{\zeta}[f,A_N]\zeta(s)\;\bar{\zeta}[f,A_M]\zeta(t)\;Y^N(s)Y^M(t)\;q(s,t)dsdt\,. \end{aligned} \tag{3.11}$$

From this expression we can read off directly the vertices for a perturbative evaluation of $\hat{\mathcal{Z}}$. Obviously, due to the triviality of our target space fields (G,B,Φ), there are no vertices in the bulk of the string world sheet, but on the boundary only. The propagator $\langle Y^M(s)Y^N(t)\rangle$ restricted to the boundary is equal to $-2\alpha'\eta^{MN}\log|s-t|$. Since this is integrable, in lowest order the only divergences are due to the tadpole diagrams arising by connecting the Y-legs of either the third or the fifth vertex in (3.11). The divergence arising from the third term contains $\dot{X}^M$, hence it contributes to the gauge field β-function. The fifth term contains $\partial_{\mathbf{n}}X^1 = \int\bar{\zeta}f\zeta q(s,t)dt$. This is an independent divergence [9, 13] and constitutes the β-function for the brane position f. Altogether we find

$$\begin{aligned}\beta_N^{(A)} &= -\alpha'\,D^MF_{MN}\\ \beta^{(f)} &= \frac{i}{2\pi}D^M[f,A_M] = \frac{1}{2\pi}D^MD_Mf\,. \end{aligned} \tag{3.12}$$

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4. β-functions in the case of dynamically realized Dirichlet boundary condition

In this section we consider a model defined by a functional integral without any constraint on the functional integrand and take as the action that which arises by formulating the T-duality as a canonical transformation [7]. Introducing

$$\breve{W}[X,\zeta,\bar{\zeta}] = \exp i\int_{\partial M}\left(\bar{\zeta}A_N\zeta\ \dot{X}^N + \frac{1}{2\pi\alpha'}\bar{\zeta}f\zeta\ \partial_{\mathbf{n}}X_1\right)ds\,, \tag{4.1}$$

the partition function is given by

$$\begin{aligned}\breve{\mathcal{Z}}[A,f] &= \int DX^\nu\ D\bar{\zeta}D\zeta\ \bar{\zeta}_a(0)\zeta_a(1)\ e^{iS_0[\zeta,\bar{\zeta}]}\times\\ &\times\ \exp\left(\frac{i}{4\pi\alpha'}\int_M d^2z\ \partial X^\mu\partial X_\mu - \frac{i}{2\pi\alpha'}\int_{\partial M}X^1\partial_{\mathbf{n}}X_1 ds\right)\breve{W}[X,\zeta,\bar{\zeta}]\,.\end{aligned} \tag{4.2}$$

The Dirichlet condition arises as part of the stationarity condition of the action since $X^1 - \bar{\zeta}f\zeta$ is the factor multiplying $\partial_{\mathbf{n}}\delta X_1$ on ∂M. In contrast to the Wilson loops of the previous sections $\breve{W}$ is, due to the presence of $\partial_{\mathbf{n}}X_1$, a functional of $X^\mu(z)$ not only on the boundary, but on the string world sheet M itself. Again, after a partial integration, we get as a starting point for the variational formulae

$$\begin{aligned}\frac{\delta\breve{W}}{\delta X^N(z)} &= i\ \breve{W}\int_0^1 \delta^{(2)}(z(s)-z)\times\\ &\times\left(\bar{\zeta}(\partial_N A_K - \partial_K A_N)\zeta\dot{X}^K + \frac{1}{2\pi\alpha'}\bar{\zeta}\partial_N f\zeta\partial_{\mathbf{n}}X_1 - \dot{\bar{\zeta}}A_N\zeta - \bar{\zeta}A_N\dot{\zeta}\right)ds\,.\end{aligned} \tag{4.3}$$

The total action for $\zeta, \bar{\zeta}$ is

$$\begin{aligned}\breve{S}[\zeta,\bar{\zeta}] &= S_0[\zeta,\bar{\zeta}] - i\log\breve{W}\\ &= S_0[\zeta,\bar{\zeta}] + \int_0^1 \bar{\zeta}A_N\zeta\dot{X}^N ds + \frac{1}{2\pi\alpha'}\int_{\partial M}\bar{\zeta}f\zeta\partial_{\mathbf{n}}X_1 ds\,.\end{aligned} \tag{4.4}$$

With

$$\dot{\zeta}(t) = iA_N\zeta\dot{X}^N(t) + \frac{i}{2\pi\alpha'}f\zeta(t)\partial_{\mathbf{n}}X_1 - \ i\frac{\delta\breve{S}}{\delta\bar{\zeta}(s)} \tag{4.5}$$

we repeat the procedure of section 2 and get

$$\frac{\delta\breve{W}}{\delta X^N(z)} = i\breve{W}\int_0^1 \delta^{(2)}(z(s)-z)\left(\bar{\zeta}F_{NK}\zeta\ \dot{X}^K + \frac{1}{2\pi\alpha'}\bar{\zeta}D_N f\zeta\ \partial_{\mathbf{n}}X_1\right)ds\,, \tag{4.6}$$

$$\begin{aligned}\frac{\delta^2\breve{W}}{\delta X^N(z)\delta X^M(z')} &= -\breve{W}\int_0^1 \delta^{(2)}(z(s)-z)\left(\bar{\zeta}F_{NK}\zeta\dot{X}^K + \frac{1}{2\pi\alpha'}\bar{\zeta}D_N f\zeta\partial_{\mathbf{n}}X_1\right)ds\ \times\\ &\times\int_0^1 \delta^{(2)}(z(t)-z')\left(\bar{\zeta}F_{ML}\zeta\dot{X}^L + \frac{1}{2\pi\alpha'}\bar{\zeta}D_M f\zeta\partial_{\mathbf{n}}X_1\right)dt\\ &+i\breve{W}\int\delta^{(2)}(z(s)-z')\delta^{(2)}(z(s)-z)\left(\bar{\zeta}D_M F_{NK}\zeta\dot{X}^K + \frac{1}{2\pi\alpha'}\bar{\zeta}D_M D_N f\zeta\partial_{\mathbf{n}}X_1\right)ds\\ &+i\breve{W}\int\bar{\zeta}F_{NM}\zeta(s)\ \delta^{(2)}(z(s)-z)\ \frac{d}{ds}\delta^{(2)}(z(s)-z')ds\,.\end{aligned} \tag{4.7}$$

The variations with respect to X^1 are different qualitatively. There is no way to get rid of $\partial_{\mathbf{n}}$ acting on the variation of X^1 by some partial integration on the boundary. Thus no $\dot\zeta$ or $\dot{\bar\zeta}$ is generated, and we have no opportunity to apply the equations of motion for $\zeta, \bar\zeta$ in a useful manner.[5] The first and second derivatives are

$$\begin{aligned}
\frac{\delta \breve{W}}{\delta X_1(z)} &= \frac{i\breve{W}}{2\pi\alpha'} \int \bar\zeta f \zeta \; \partial_{\mathbf{n}} \delta^{(2)}(z(s)-z) ds \,, \\
\frac{\delta^2 \breve{W}}{\delta X_1(z)\delta X_1(z')} &= -\frac{\breve{W}}{(2\pi\alpha')^2} \int \bar\zeta f \zeta \; \partial_{\mathbf{n}} \delta^{(2)}(z(s)-z) ds \int \bar\zeta f \zeta \; \partial_{\mathbf{n}} \delta^{(2)}(z(t)-z') dt.
\end{aligned} \quad (4.8)$$

There is still a mixed derivative $\frac{\delta^2 \breve{W}}{\delta X^1(z) \delta X^N(z')}$, but we save writing down its form and proceed to the combination of all first and second derivatives in

$$\breve{W}[X+Y,\zeta,\bar\zeta] = \breve{W}[X,\zeta,\bar\zeta] \;\; \exp(i\breve\Omega[X,Y,\zeta,\bar\zeta])\,, \quad (4.9)$$

with

$$\begin{aligned}
\breve\Omega[X,Y,\zeta,\bar\zeta] = \int_0^1 \Big\{ & \Big(\bar\zeta F_{NL}\zeta(s) \; \dot X^L + \frac{1}{2\pi\alpha'}\bar\zeta D_N f\zeta \; \partial_{\mathbf{n}} X_1\Big) Y^N \\
& + \frac{1}{2\pi\alpha'}\bar\zeta f\zeta \; \partial_{\mathbf{n}} Y_1 \; + \; \frac{1}{2}\bar\zeta F_{NM}\zeta \; \dot Y^M Y^N \\
& + \frac{1}{2}\Big(\bar\zeta D_M F_{NK}\zeta \; \dot X^K + \frac{1}{2\pi\alpha'}\bar\zeta D_M D_N f\zeta \; \partial_{\mathbf{n}} X_1\Big) Y^M Y^N \\
& + \frac{1}{4\pi\alpha'}\bar\zeta D_M f\zeta \; Y^M \partial_{\mathbf{n}} Y_1 + O(Y^3)\Big\} ds\,.
\end{aligned} \quad (4.10)$$

To calculate the partition function (4.2) we make a shift $X \to X+Y$, use $D(X+Y) = DY$ and choose X in such a way that the linear terms in Y vanish in the bulk of M, i.e. $\partial^2 X^\mu = 0$. However, we do not specify any boundary condition on X^μ.[6] Then we have to continue with

$$\begin{aligned}
\breve{\mathcal{Z}} = & \int D\bar\zeta D\zeta \; \bar\zeta_a(0)\zeta_a(1) \; e^{iS_0[\zeta,\bar\zeta] + \frac{i}{4\pi\alpha'}\int (X^N\partial_{\mathbf{n}} X_N - X^1 \partial_{\mathbf{n}} X_1) ds} \; \breve{W}[X,\zeta,\bar\zeta] \times \\
& \times \int DY^\nu \;\; \exp\Big\{\frac{i}{4\pi\alpha'}\Big(\int \partial Y^\mu \partial Y_\mu d^2z - 2\int Y^1 \partial_{\mathbf{n}} Y_1 ds\Big)\Big\} \times \\
& \times \exp\Big\{\frac{i}{2\pi\alpha'}\int (Y_N \partial_{\mathbf{n}} X^N - X^1 \partial_{\mathbf{n}} Y_1) ds \; + \; i\breve\Omega[X,Y,\zeta,\bar\zeta]\Big\}.
\end{aligned} \quad (4.11)$$

For the perturbative evaluation of the Y-integral it is convenient to use

$$\begin{aligned}
& \int DY^\nu \;\; \exp\Big\{\frac{i}{4\pi\alpha'}\int \partial Y^\mu \partial Y_\mu \; d^2z - \frac{i}{2\pi\alpha'}\int Y^1 \partial_{\mathbf{n}} Y_1 + i\int Y^\mu J_\mu\Big\} = \\
& = (\det\Delta)^{\frac{1}{2}} \exp\Big\{\frac{-i}{2}\int J^\alpha(z)\Delta_{\alpha\beta}(z,z') J^\beta(z') d^2z d^2z'\Big\}.
\end{aligned} \quad (4.12)$$

[5]Fortunately here these manipulations are not needed to get manifest gauge invariant structures.

[6]We could choose X in such a way that all linear terms in Y would vanish, but then the factorizing classical factor would contain $\zeta, \bar\zeta$ and mix with $\breve{W}[X,\zeta,\bar\zeta]$ and the Y functional integral in doing the ζ-integration.

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The propagator Δ of the above equation is defined by[7]

$$\begin{aligned}\Delta_{\mu\nu} &= \eta_{\mu\nu}\Delta_\nu \qquad (\text{no sum})\,,\\ \partial^2\Delta_{\mu\nu} &= -2\pi\alpha'\eta_{\mu\nu}\delta^{(2)}(z-z') \quad \text{ in } M, \quad \Delta_1 = 0\,, \ \ \partial_{\mathbf{n}}\Delta_N = 0 \quad \text{on } \partial M\,. \end{aligned} \tag{4.13}$$

Equations (4.11)-(4.13) imply a set of Feynman rules with Neumann propagator for Y^N, Dirichlet propagator for Y^1, no vertex in the bulk and boundary vertices defined by $\breve{\Omega}$ as well as the two additional 1-leg vertices in the last exponential of eq.(4.11).

The two vertices in the third line of eq.(4.10) yield divergent tadpole contributions which would deliver just the same β-functions as in section 3, if they would be the only divergent diagrams. But in contrast to section 3, where 1-leg vertices appeared with underivated Y only, now we have the 1-leg vertex $\frac{i}{2\pi\alpha'}(\bar\zeta f\zeta - X^1)\partial_{\mathbf{n}}Y_1$. The diagram connecting two such 1-leg vertices is

$$\mathcal{A} \equiv \frac{1}{2}\,\frac{-1}{(2\pi\alpha')^2}\int ds dt\ (\bar\zeta f\zeta(s) - X^1)(\bar\zeta f\zeta(t) - X^1)\ \partial_{\mathbf{n}}\partial'_{\mathbf{n}}\Delta_1(z,z')|_{z=s,\ z'=t}\,. \tag{4.14}$$

For the upper half plane

$$\Delta_1(z,z') = -\alpha'\ (\log|z-z'| \ - \ \log|z-\bar z'|) \tag{4.15}$$

leads to

$$\mathcal{A} = \frac{-1}{4\pi^2\alpha'}\int ds dt\ \frac{(\bar\zeta f\zeta(s) - X^1)(\bar\zeta f\zeta(t) - X^1)}{(s-t)^2}\,. \tag{4.16}$$

$\mathcal{A}$ is linearly divergent. We neglect linear divergences[8] and look only for logarithmic ones which could be produced by expanding the nominator in $\mathcal{A}$. If this nominator would be continuously differentiable, due to the antisymmetry of $(s-t)^{-1}$, there would be no such divergence. However, we have to evaluate the diagram with quantised fields Y *and* $\zeta, \bar\zeta$. As a consequence the left and right limits of the derivative of the nominator could differ. Anticipating this possibility we expand the nominator for $s > t$ and $s < t$ separately

$$\begin{aligned}\bar\zeta f\zeta(s)\ \bar\zeta f\zeta(t) = O(1) \ + \ (s-t)\Big\{&\Theta(s-t)\ \frac{d}{dt}\Big(\bar\zeta f\zeta(t+0)\Big)\,\bar\zeta f\zeta(t)\\ &+\Theta(t-s)\ \frac{d}{dt}\Big(\bar\zeta f\zeta(t-0)\Big)\,\bar\zeta f\zeta(t)\Big\} \ + \ O((s-t)^2)\,. \end{aligned}\tag{4.17}$$

Using the equations of motion (4.5) this turns into

$$\begin{aligned}\bar\zeta f\zeta(s)\ \bar\zeta f\zeta(t) = O(1) \ + \ (s-t)\ \dot X^N\ \Big\{&\Theta(s-t)\ \bar\zeta D_N f\zeta(t+0)\ \bar\zeta f\zeta(t)\\ &+\Theta(t-s)\ \bar\zeta D_N f\zeta(t-0)\ \bar\zeta f\zeta(t)\Big\} \ + \ O((s-t)^2)\,. \end{aligned}\tag{4.18}$$

[7]To avoid the discussion of the zero mode for the Neumann propagator we take M to be the upper half plane.

[8]They are irrelevant in dimensional regularisation.

Now from the special structure of the ζ-propagator it is obvious that for the purpose of the perturbative evaluation of the $\zeta, \bar{\zeta}$ functional integral we can make the following identifications

$$\begin{aligned}\bar{\zeta}_a(t+0)\ \zeta_b(t+0)\ \bar{\zeta}_c(t)\ \zeta_d(t) &= \delta_{bc}\ \bar{\zeta}_a(t+0)\ \zeta_d(t)\,,\\ \bar{\zeta}_a(t-0)\ \zeta_b(t-0)\ \bar{\zeta}_c(t)\ \zeta_d(t) &= \delta_{ad}\ \bar{\zeta}_c(t)\ \zeta_b(t-0)\,.\end{aligned} \tag{4.19}$$

Using this in (4.18) we get for the regularised version (a regul. parameter) of (4.16)[9]

$$\begin{aligned}\mathcal{A} = \frac{-1}{4\pi^2\alpha'}\int \Big(\bar{\zeta}(D_N f) f \zeta(t) \int_{s>t} \frac{s-t}{(s-t)^2+a^2}\ ds\\ + \bar{\zeta} f D_N f \zeta(t) \int_{s<t} \frac{s-t}{(s-t)^2+a^2}\ ds\Big) \dot{X}^N(t) dt\ +\ \ldots\,,\end{aligned}$$

the dots standing for either linear divergent or finite contributions. This implies

$$\mathcal{A} = \frac{-1}{4\pi^2\alpha'} \log a \int \dot{X}^N \bar{\zeta}\ [f, D_N f]\ \zeta(t) dt\ +\ \ldots\,. \tag{4.20}$$

This divergence is responsible for the presence of the mass term in the gauge field β-function. Altogether we find

$$\beta_N^{(A)} = -\alpha'\ D^M F_{MN}\ +\ \frac{i}{4\pi^2\alpha'}[f, D_N f]\,. \tag{4.21}$$

$\beta^{(f)}$ coincides with that of section 3.

5. Conclusions

At least up to the considered order of perturbation theory, our results in section 3 and 4 clearly favour the D-brane model with dynamically implemented Dirichlet boundary conditions as the correct realization of the T-dual partner of the open string with free ends. The model with externally realized boundary conditions has been motivated [7, 13] by formal manipulations of the functional integral. Its failure to produce the mass term can be explained just by this formal nature. However, on a more ambitious level we would like to localise the forbidden step in those manipulations. A first idea we get by looking back to the application of the naive T-duality rules to the gauge field β-function of the open string with free ends. The mass term arises from taking the value 1 for the summation index. The underlying ultraviolet divergence is due to a propagator in 1-direction at coinciding points. Since there is no propagation in this direction in the model of section 3, it should be not surprising that no mass term appears. On the more technical level the comparison of the details of the calculations is useful.

[9] The terms in the nominator of (4.16) containing f in zero or first order are continuously differentiable.

In both cases we consider the partition functions $\hat{\mathcal{Z}}$, $\breve{\mathcal{Z}}$ as defined by the interacting quantum field theory of the string world sheet position X and the auxiliary fields $\zeta, \bar{\zeta}$ in the sense of a combined functional integral. This point of view is implemented by the use of the ζ equations of motion including contact terms for the expansion of the modified Wilson loops.[10] However, we have given up this understanding at one occasion, namely when we switched to a partially stepwise evaluation by performing the X^1 integration in section 3. But just the intermediate treatment of $\zeta, \bar{\zeta}$ as classical fields is the dangerous step as can be seen in the calculation of diagram $\mathcal{A}$ in section 4. If we there had treated $\zeta, \bar{\zeta}$ as classical fields we would also have lost the mass term.

The stepwise, i.e. first X and then $\zeta, \bar{\zeta}$ integration has been crucial for giving the formal notion of matrix valued brane position a technical meaning via insertion of matrices in the argument which specifies an explicit boundary parameter dependent scalar valued Dirichlet boundary condition [7, 13]. Now we see that just this stepwise procedure violates the equivalence with the original free end model. Nevertheless the model of [13] is well defined and interesting for its own.

We stress that the preference of the dynamically over the external realization of boundary conditions developed in this paper concerns the case of non-Abelian gauge fields and the implementation of path ordering via the ζ-formalism only. In the Abelian case we see no stumbling block in the equivalent transformation of the free end model to the D-brane model with externally realized boundary conditions. We can also not exclude that in the non-Abelian case there could exist T-dual versions with external boundary conditions beyond the ζ-formalism, for some alternative aspects of the construction of a σ-model describing matrix D-branes see [18].

Our discussion concerned the test of quantum equivalence of naive T-dual models in the general, not necessary conformally invariant situation. Since T-duality as a map between equivalent theories is by far better established in the conformal case, it would be interesting to understand both the potential differences between externally and dynamically realized boundary conditions as well as the σ-model description of boundaries coupling to non-Abelian gauge fields in the formalism of boundary conformal field theories, [19] and refs. therein.

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[10]Note that in *both* cases the ζ equations of motion are compatible with naive duality.

References

[1] J. Dai, R.G. Leigh, J. Polchinski, *Mod. Phys. Lett.* **A 4** (1989) 2073;
P. Horava, *Phys. Lett.* **B 231** (1989) 251;
M.B. Green, *Phys. Lett.* **B 266** (1991) 325.

[2] J. Polchinski, S. Chaudhuri, C.V. Johnson, preprint *Notes on D-branes*, `hep-th/9602052`;
J. Polchinski, preprint *TASI lectures on D-branes*, `hep-th/9611050`.

[3] E. Witten, *Nucl. Phys.* **B 460** (1996) 335, `hep-th/9510135`.

[4] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, *Phys. Rev.* **D 55** (1997) 5112, `hep-th/9610043`.

[5] M.R. Douglas, preprint *D-branes and matrix theory in curved space*, `hep-th/9707228`.

[6] T.H. Buscher, *Phys. Lett.* **B 194** (1987) 51; *ibid.* **201** (1988) 466;
A. Giveon, M. Poratti, E. Rabinovici, *Phys. Rep.* **244** (1994) 77.

[7] H. Dorn, H.-J. Otto, *Phys. Lett.* **B 381** (1996) 81 [`hep-th/9603186`]; *Nucl. Phys.* **56** *(Proc. Suppl.)* (1997) 30 [`hep-th/9702018`].

[8] A.A. Tseytlin, *Nucl. Phys.* **B 469** (1996) 51 [`hep-th/9602064`];
E. Alvarez, J.L.F. Barbon, J. Borlaf, *Nucl. Phys.* **B 479** (1996) 218 [`hep-th/9603089`];
E. Bergshoeff, M. De Roo, *Phys. Lett.* **B 380** (1996) 265 [`hep-th/9603123`];
S. Förste, A.A. Kehagias, S. Schwager, *Nucl. Phys.* **B 478** (1996) 141 [`hep-th/9604013`].

[9] R.G. Leigh, *Mod. Phys. Lett.* **A 4** (1989) 2767.

[10] A.A. Tseytlin, *Mod. Phys. Lett.* **A 6** (1991) 1721;
J. Balog, P. Forgacs, Z. Horvath, L. Palla, *Phys. Lett.* **B 388** (1996) ??????? [`hep-th/9606187`];
P. Haagensen, *Phys. Lett.* **B 383** (1996) 356 [`hep-th/9604136`];
N. Kaloper, K.A. Meissner, *Phys. Rev.* **D 56** (1997) 7940 [`hep-th/9705193`].

[11] S. Samuel, *Nucl. Phys.* **B 149** (1979) 517;
R.A. Brandt, F. Neri, D. Zwanziger, *Phys. Rev.* **D 19** (1979) 1153;
I. Ya. Arefyeva, *Phys. Lett.* **B 93** (1980) 347;
H. Dorn, *Fortschr. d. Phys.* **34** (1986) 11.

[12] J.L. Gervais, A. Neveu, *Nucl. Phys.* **B 163** (1980) 189.

[13] H. Dorn, *Nucl. Phys.* **B 494** (1997) 105 [`hep-th/9612120`].

[14] H. Dorn, preprint *Nonabelian gauge field dynamics on matrix D-branes in curved space and two-dimensional σ-models*, `hep-th/9712057`

[15] H. Dorn, H.-J. Otto, *Z. Physik* **C 32** (1986) 599.

[16] D. Brecher, M.J. Perry, preprint *Bound states of D-branes and the Non-Abelian Born-Infeld action*, `hep-th/9801127`.

[17] S. Mandelstam, *Phys. Rev.* **D 19** (1979) 2391;
Y. Nambu, *Phys. Lett.* **B 80** (1979) 372;
A.M. Polyakov, *Phys. Lett.* **B 82** (1979) 247; *Nucl. Phys.* **B 164** (1980) 171.

[18] F. Lizzi, N.E. Mavromatos, R.J. Szabo, preprint *Matrix σ-models for multi D-brane dynamics*, `hep-th/9711012`

[19] A. Recknagel, V. Schomerus, preprint *D-branes in Gepner models*, `hep-th/9712186`;
J. Fuchs, C. Schweigert, preprint *Branes: from free fields to general backgrounds*, `hep-th/9712257`.